Exact asymptotic results for persistence in the Sinai model with arbitrary drift

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We obtain exact asymptotic results for the disorder averaged persistence of a Brownian particle moving in a biased Sinai landscape. We employ a method that maps the problem of computing the persistence to the problem of finding the energy spectrum of a single-particle quantum Hamiltonian, which can be subsequently found. Our method allows us analytical access to arbitrary values of the drift (bias), thus going beyond the previous methods that provide results only in the limit of vanishing drift. We show that on varying the drift the persistence displays a variety of rich asymptotic behaviors including, in particular, interesting qualitative changes at some special values of the drift.

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I. INTRODUCTION

Persistence, i.e., the probability that a fluctuating field does not change sign upto time t has been widely studied in recent years in the context of nonequilibrium systems [1]. A wide variety of results, both theoretical and experimental, are available for pure systems. In contrast, there have been very few studies of persistence in disordered systems. A notable exception is the study of persistence, theoretical [2] as well as numerical [3], in disordered Ising models. In this paper we study analytically the persistence in another disordered system, namely, the celebrated Sinai model [4], but in the presence of an additional arbitrary drift. We show that as one varies the drift parameter, the disorder averaged persistence displays a wide variety of rich behaviors that undergo qualitative changes at certain special values of the drift.

The Sinai model [4] is perhaps one of the simplest models of disordered systems where various disorder averaged physical quantities exhibit rich and nontrivial behaviors and yet, can be computed analytically [5]. Thus the Sinai model serves the role of the ''Ising'' model in disordered systems. In this model a Brownian particle undergoes diffusion in the presence of a random time-independent potential. The position $x(t)$ of the particle evolves via the Langevin equation

$$\frac{dx}{dt} = -\frac{dU}{dx} + \eta(t), \tag{1}$$

where $\eta(t)$ is the thermal noise with $\langle \eta(t) \rangle = 0$ and $\langle \eta(t)\,\eta(t') \rangle = \delta(t-t')$ and $U(x)$ is the external potential. In the biased Sinai model one considers the potential to be simply $U(x) = -\mu x + \sqrt{\sigma} B(x)$, where $B(x)$ represents the trajectory of a Brownian motion in space, i.e., $B(x) = \int_0^x \xi(x')dx'$ with $\langle \xi(x) \rangle = 0$ and $\langle \xi(x)\xi(x') \rangle = \delta(x-x')$. The parameter μ represents the bias or the drift and σ measures the strength of the disorder. Thus the particle is subjected, in addition to the thermal noise $\eta(t)$, to an external position dependent random force $F(x) = -dU/dx = \mu + \xi(x)$. Various physical quantities in the Sinai model have been studied before [5]. Here our aim is to compute the persistence $P(x_0,t)$ defined as the probability that the particle does not cross the origin upto time t starting at the initial position $x_0 \geqslant 0$ at $t=0$. Evidently this quantity will depend on the realization of the underlying disorder potential and will, in general, vary from one sample of disorder to another. Our final goal is to compute the disorder averaged persistence $\overline{P(x_0,t)}$ as a function of both x_0 and t for different values of the drift μ.

Physicists have recently studied the persistence in the Sinai model using various methods that include an exact probabilistic approach suited for unbiased Sinai model [6], a study of an equivalent lattice model with random hopping rates [7], and also by employing a real space renormalization group method [8]. All of these methods provide asymptotically exact results, but *only in the limit of vanishing drift*, i.e., when $\mu \to 0$. Unfortunately, extension of these existing physical methods to extract explicit asymptotic results for arbitrary nonzero μ seems rather difficult. Mathematicians, on the other hand, have studied some aspects of a related quantity namely the distribution of the first-passage time in the Sinai model for nonzero drift and some rigorous results seem to exist [9–12]. However, these mathematical methods are extremely technical and difficult to follow. What is lacking, so far, is a unified physical approach that, besides reproducing the known results in a simple and transparent way, provides exact asymptotic results for all μ and yet simple and powerful enough to be easily generalizable to other problems. The purpose of this paper is to provide such an approach. The heart of our approach lies in mapping the problem of computing the persistence in the Sinai model with arbitrary drift to finding the spectrum of a single-particle quantum Hamiltonian, which can subsequently be done exactly.

Apart from presenting a unified approach valid for arbitrary drift, there are two other physical motivations for this work. First, it is well known that the Sinai model displays a range of intereresting anomalous diffusion properties as one tunes the drift μ through certain finite ''critical'' values [5]. It is therefore theoretically interesting to know how the persistence behavior changes as the drift is varied through these critical values. Second, and perhaps more importantly, the Sinai model with a nonzero drift has numerous physical applications [5] including the diffusion of electrons in a disor-

dered medium in the presence of an electric field, glassy dynamics of dislocations in solids, dynamics of random field magnets, and dynamics near the helix-coil transitions in heteroploymers. The most recent application of the biased Sinai model has been to understand the dynamics of denaturation of a single DNA molecule under an external force [13]. Persistence seems to be a natural quantity to study in these systems and hence we expect that the analytical results presented in this paper will be useful in many of the physical situations mentioned above.

The paper is organized as follows. In Sec. II, we present our general approach. A detailed discussion of the pure case ($\sigma=0$) with a nonzero drift is presented in Sec. II A which will help us anticipate the general features of persistence in the disordered case studied later in Sec. II B where we illustrate the mapping to a quantum mechanics problem. In Sec. III, we discuss the results for the disorder averaged persistence for positive drift ($\mu>0$). The results for the negative drift ($\mu<0$), fundamentally different from the positive drift case, are detailed in Sec. IV. We conclude in Sec. V with a summary and outlook. The details of the derivation of the eigenvalue spectrum of the quantum Hamiltonian are presented in the Appendix A. In Appendix B, we present an alternative derivation of the disordered averaged persistence in the case of positive drift. The details of the second-order perturbation theory for negative drift are presented in Appendix C.

II. GENERAL APPROACH

Consider the particle whose position $x(t)$ evolves via the Langevin equation (1) starting initially at $x(t=0)=x_0$. The persistence $P(x_0,t)$ is the probability that the particle does not cross the origin up to time t starting at x_0. It is also useful to define the distribution of the first-passage time $F(x_0,t_*)$, which is simply the probability that the particle hits the origin for the first time at $t=t_*$ starting initially at x_0. The distribution $F(x_0,t_*)$ is related to the persistence $P(x_0,t)$ via the simple relation $P(x_0,t)=1-\int_0^t F(x_0,t_*)dt_*$. This follows from the fact that the integral $\int_0^t F(x_0,t_*)dt_*$ sums up the probabilities of all the events when the particle hits the origin before time t and when subtracted from 1 (which is the total probability), the resulting quantity, by definition, is the persistence. Our objective is to first compute $F(x_0,t_*)$ or its Laplace transform and then use the above relationship to compute the persistence $P(x_0,t)$.

In order to calculate the first-passage time distribution $F(x_0,t_*)$ we employ a powerful backward Fokker-Planck approach that has been used before to study the persistence in the unbiased Sinai model [6] as well as in other contexts [14]. It is instructive to start with a more general quantity $Q_p(x_0)=\langle \exp(-p\int_0^{t*} V[x(t')]dt')\rangle_{x_0}$, where $\langle\rangle$ denotes the thermal average, $V[x(t)]$ is an arbitrary functional and t_* denotes the first-passage time, i.e., the time at which the particle first hits the origin starting initially at x_0 at $t=0$. Note that if we choose $V(x)=1$, then $Q_p(x_0)=\langle e^{-pt_*}\rangle_{x_0}=\int_0^\infty e^{-pt_*}F(x_0,t_*)dt_*$ is simply the Laplace transform of the first-passage time distribution $F(x_0,t_*)$. For the convenience of notations, we will henceforth denote the initial position $x_0=x$ and the first-passage time $t_*=t$.

A differential equation for $Q_p(x)$ can be derived by evolving the particle from its initial position x over an infinitesimal time dt. This gives $Q_p(x)=\langle(1-pVdt)Q_p(x+dx)\rangle$, where dx is the displacement of the particle in time dt from its initial position x. Using Eq. (1) one gets $dx=F(x)dt+\eta(0)dt$, where $F(x)=-dU/dx=\mu+\sqrt{\sigma}\xi(x)$ is the random force. Expanding $Q_p[x+F(x)dt+\eta(0)dt]$ to order dt and averaging over $\eta(0)$, one arrives at the backward Fokker-Planck equation,

$$\frac{1}{2}\frac{d^2Q_p}{dx^2}+F(x)\frac{dQ_p}{dx}-pV(x)Q_p=0. \tag{2}$$

Since here we are interested only in the first-passage time distribution, we will henceforth set $V(x)=1$. Note, however, that this method is powerful enough to deal with the statistical properties of any arbitrary functional $V(x)$ of the stochastic process. We will also assume, without any loss of generality, that the initial position $x\geqslant 0$. For $x\leqslant 0$, one will obtain the same results by changing the sign of the drift μ. Eq. (2) is supplemented with the two boundary conditions: (i) $Q_p(x=0)=1$, since if the particle starts at $x=0$, obviously its first-passage time $t=0$ and (ii) $Q_p(x\to\infty)=0$, since the first-passage time $t\to\infty$ if the particle starts at $x=\infty$.

In Secs. II A and II B we discuss the solutions of the differential equation (2), respectively, for the pure case ($\sigma=0$) and the disordered case ($\sigma>0$).

A. Pure case with nonzero drift

It is instructive to discuss first the pure case with nonzero drift ($\mu\neq 0$) in the absence of the random potential ($\sigma=0$). The results for the pure case will help us anticipate what to expect for the disordered case later. Solving Eq. (2) with $F(x)=\mu$ we get $Q_p(x)=\exp(-[\mu+\sqrt{\mu^2+2p}]x)$ that satisfies the required boundary conditions. Since $Q_p(x)=\int_0^\infty e^{-pt}F(x,t)dt$, we need to invert the Laplace transform that gives

$$F(x,t)=x(2\pi t^3)^{-1/2}e^{-(x+\mu t)^2/2t}. \tag{3}$$

Using $P(x,t)=1-\int_0^t F(x,t')dt'$, one then gets

$$P(x,t)=1-\frac{x}{\sqrt{2\pi}}\int_0^t t'^{-3/2}e^{-(x+\mu t')^2/2t'}dt'. \tag{4}$$

Let us analyze what happens for large t in the three separate cases (i) $\mu>0$, (ii) $\mu<0$, and (iii) $\mu=0$.

(i) For positive bias away from the origin ($\mu>0$), the particle eventually escapes to ∞ with a nonzero probability and $P(x,t)\to P(x)$ as $t\to\infty$. Taking $t\to\infty$ limit in Eq. (4) one easily obtains this eventual persistence "profile" $P(x)=1-e^{-2\mu x}$.

(ii) In the opposite case ($\mu=-|\mu|<0$), it follows from Eq. (4) that as $t\to\infty$, $P(x,t)\approx\sqrt{2/\pi}x|\mu|^{-2}t^{-3/2}e^{-(x-|\mu|t)^2/2t}$. In this case, a more useful

information is contained in the asymptotic first-passage distribution $F(x,t)$. From the exact expression of $F(x,t)$ in Eq. (3), we find that in the appropriate scaling limit $x\to\infty$, $t\to\infty$ but keeping x/t fixed, the first-passage distribution approaches a delta function, $F(x,t)\to\delta(t-x/|\mu|)$. Equivalently the Laplace transform $Q_p(x)\to e^{-px/|\mu|}$, which also follows directly from the expression of $Q_p(x)=\exp[-(\sqrt{|\mu|^2+2p}-|\mu|)x]$ in the correct scaling limit $x\to 0$, $p\to 0$ but keeping the product px fixed. Thus in this case, at late times, the particle essentially moves ballistically with velocity $|\mu|$ and crosses the origin for the first time at $t=x/|\mu|$.

(iii) In the unbiased case ($\mu=0$), we recover from Eq. (4) the well known exact result [15], $P(x,t)=\mathrm{erf}(x/\sqrt{2t})$, where $\mathrm{erf}(x)=(2/\sqrt{\pi})\int_0^x e^{-u^2}du$ is the error function.

In the following section, we switch on the quenched disorder ($\sigma>0$) and examine the consequences on the asymptotic properties of the disorder averaged persistence.

B. Disordered case with nonzero drift

Unlike the pure case, we can no longer solve the differential equation (2) exactly for the disordered case since $F(x)=\mu+\sqrt{\sigma}\xi(x)$ now has an x-dependent random part. To make further progress we first make a Hopf-Cole transformation, $Q_p(x)=\exp[-\int_0^x z_p(x')dx']$. Clearly $z_p(x)=-d\ln Q_p(x)/dx$ is a slope variable. By construction, $Q_p(x)$ automatically satisfies the boundary condition $Q_p(x=0)=1$. Substituting this form of $Q_p(x)$ in Eq. (2), we find that the slope variable $z_p(x)$ satisfies a first-order stochastic Riccati equation

$$\frac{dz_p(x')}{dx'}=z_p^2(x')-2[\mu+\sqrt{\sigma}\xi(x')]z_p(x')-2p. \qquad (5)$$

The right hand side of the above equation contains a multiplicative noise term and we will interpret it according to the Stratonovich prescription. Note that since Eq. (5) is a first order equation, $z_p(x')$ at an arbitrary x' will be fully determined as a functional of the noise history $\{\xi(x')\}$, at least in principle, provided the value of z_p is known at some ''initial'' point. Note that this initial point can be anywhere. The program would then be to substitute this fully determined functional to evaluate the integral $\int_0^x z_p(x')dx'$ and thereby determine $Q_p(x)=\exp[-\int_0^x z_p(x')dx']$. Subsequently one would perform the disorder average $\overline{Q_p(x)}$ where the overbar indicates an average over the noise history $\{\xi(x')\}$ for fixed x.

However, there is one problem in implementing this program, namely, the initial value of $z_p(x')$ is not specified. Consequently the solution of the first-order equation (5) will involve an unknown parameter, i.e., the initial value of $z_p(x')$. There exists, however, a rather nice trick to get around this difficulty. This trick has been used before in the Sinai model in various contexts [5,6,16]. It is useful to outline this trick in the present context. To use this trick, we first fix x in the definition $Q_p(x)=\exp[-w(x)]$ where we have defined $w(x)=\int_0^x z_p(x')dx'$. Keeping x fixed we then make

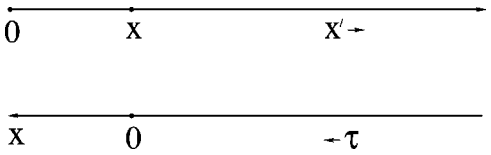

FIG. 1. The change of variable $\tau=x-x'$ for fixed x. The new variable τ increases from $\tau=-\infty$ (when $x'=\infty$) to $\tau=x$ (when $x'=0$) through the point $\tau=0$ (when $x'=x$).

a change of variable, $\tau=x-x'$. Thus, when $x'\to\infty$, $\tau\to-\infty$ and when $x'\to 0$, $\tau\to x$. Besides $x'=x$ corresponds to $\tau=0$ (see Fig. 1). Then $w(x)=\int_0^x z_p(x')dx'=-\int_x^0 z_p(x-\tau)d\tau=\int_0^x \tilde{z}_p(\tau)d\tau$, where $\tilde{z}_p(\tau)=z_p(x-\tau)$. In this new τ variable, Eq. (5) becomes

$$\frac{d\tilde{z}_p(\tau)}{d\tau}=-\tilde{z}_p^2(\tau)+2[\mu+\sqrt{\sigma}\tilde{\xi}(\tau)]\tilde{z}_p(\tau)+2p, \qquad (6)$$

where $\tilde{\xi}(\tau)=\xi(x-\tau)$ and $\tau\in[-\infty,x]$. Note that $\langle\tilde{\xi}(\tau)\rangle=0$ and $\langle\tilde{\xi}(\tau)\tilde{\xi}(\tau')\rangle=\delta(\tau-\tau')$. To simplify further, we substitute $\tilde{z}_p(\tau)=\exp[\phi(\tau)]$ in Eq. (6) and find that the variable $\phi(\tau)$ satisfies a much simplified stochastic equation containing only additive noise (and no multiplicative noise)

$$\frac{d\phi}{d\tau}=b(\phi)+2\sqrt{\sigma}\tilde{\xi}(\tau), \qquad (7)$$

where the source term $b(\phi)$ is given by

$$b(\phi)=-e^{\phi}+2\mu+2p\,e^{-\phi}. \qquad (8)$$

What did we gain in the change of variable from $x'\in[0,\infty]$ to $\tau\in[-\infty,x]$? The point is that we have, via this change of variable, gotten around the problem of initialization of the original variable $z_p(x')$ at any initial point. To see this, let us consider Eq. (7) which is valid in the regime $\tau\in[-\infty,x]$. We can interpret this equation now as a simple Langevin equation describing the evolution of the position $\phi(\tau)$ of a classical particle with ''time'' τ starting from $\tau=-\infty$. Of course, we still do not know the value of $\phi(\tau)$ at $\tau=-\infty$. The point, however, is that this initial condition at $\tau=-\infty$ is completely irrelevant. No matter what this initial condition at $\tau=-\infty$ is, it is clear from Eq. (7) that eventually when τ is far away from its starting point $\tau=-\infty$, the system will approach a stationary state. This is because the Eq. (7) describes the noisy thermal motion of a particle in a classical potential $U_{\mathrm{cl}}(\phi)=-\int_0^\phi b(u)du=e^{\phi}-2\mu\phi+2pe^{-\phi}-(2p+1)$. Hence the particle will eventually reach the equilibrium and the stationary probablity distribution of ϕ is simply given by the Gibbs measure

$$P_{\mathrm{st}}(\phi)=A\exp\left[-\frac{1}{2\sigma}U_{\mathrm{cl}}(\phi)\right]=A\exp\left[\frac{1}{2\sigma}\int_0^\phi b(u)du\right], \qquad (9)$$

where $b(\phi)$ is given by Eq. (8) and A is a normalization constant such that $\int_{-\infty}^{\infty}P_{\mathrm{st}}(\phi)d\phi=1$. For later purposes we also define $P_{\mathrm{st}}(\phi)=\psi_0^2(\phi)$, where

$$\psi_0(\phi)=\sqrt{A}\exp\left[\frac{1}{4\sigma}\int_0^{\phi}b(u)du\right]; \tag{10}$$

the function $b(\phi)$ is given by Eq. (8) and A is such that $\int_{-\infty}^{\infty}\psi_0^2(\phi)d\phi=1$.

So now we know that starting at $\tau=-\infty$ with an arbitrary initial condition, by the time the system reaches $\tau=0$, it has already achieved the stationary measure. But our task is not yet complete. We now have to evolve the system via its equation of motion (7) from $\tau=0$ to $\tau=x$ [knowing that at $\tau=0$ the distribution of ϕ is given by the Gibbs measure in Eq. (9)] and evaluate the disorder average

$$\overline{Q_p(x)}=E[\exp\{-w(x)\}]=E\left[\exp\left\{-\int_0^x e^{\phi(\tau)}d\tau\right\}\right], \tag{11}$$

where $w(\tau)=\int_0^{\tau}e^{\phi(\tau')}d\tau'$ as defined earlier and $E[\cdot]$ denotes the expectation value of the random variable inside the parenthesis. Let us introduce the quantity $R(\phi,\tau)=E_{\phi}[e^{-\lambda w(\tau)}]$ that denotes the expectation value $e^{-\lambda w}$ at time τ with $\phi(\tau)=\phi$. More precisely, if $P_J[w(\tau)=w,\phi(\tau)=\phi,\tau]$ denotes the joint probability distribution of the variables $w(\tau)$ and $\phi(\tau)$ at time τ, then $R(\phi,\tau)=\int e^{-\lambda w}P_J[w,\phi,\tau]dw$. We have introduced the additional parameter λ for later convenience, whose value will be eventually set to $\lambda=1$. Note that if we set $\lambda=0$, $R(\phi,\tau)$ is simply the probability distribution of ϕ at time τ. Thus from now on we will refer to the $\lambda=0$ case as the ''free'' problem. When $\lambda=1$, it is clear that

$$\overline{Q_p(x)}=E[\exp\{-w(x)\}]=\int_{-\infty}^{\infty}R(\phi,x)d\phi. \tag{12}$$

The advantage for this small detour in introducing the new quantity $R(\phi,\tau)$ is that one can now write down a Fokker-Planck equation for $R(\phi,\tau)$ in a straightforward manner. In fact, incrementing τ to $\tau+d\tau$ in the definition $R(\phi,\tau)=E_{\phi}[\exp\{-\lambda\int_0^{\tau}e^{\phi(\tau')}d\tau'\}]$ and using the Langevin equation (7), we find that $R(\phi,\tau)$ satisfies the following equation:

$$\frac{\partial R}{\partial\tau}=2\sigma\frac{\partial^2 R}{\partial\phi^2}-b(\phi)\frac{\partial R}{\partial\phi}-[b'(\phi)+\lambda e^{\phi}]R, \tag{13}$$

where $b'(\phi)=db/d\phi$ with $b(\phi)$ given from Eq. (8). Note that at $\tau=0$, $w(0)=0$ and hence $R(\phi,0)$ is just the probability distribution of ϕ, which is given by the Gibbs measure $R(\phi,0)=P_{\rm st}(\phi)$ in Eq. (9). Starting with this initial condition at $\tau=0$, we need to evolve Eq. (13) up to $\tau=x$, determine $R(\phi,x)$ and then integrate over ϕ in Eq. (12) to finally obtain the desired quantity $\overline{Q_p(x)}$. Note that for $\lambda=0$, Eq. (13) is simply the ordinary Fokker-Planck equation for the probability distribution of ϕ in the free problem.

We next substitute $R(\phi,\tau)=\exp[1/4\sigma\int_0^{\phi}b(u)du]\,G(\phi,\tau)$ in Eq. (13) to get rid of the first derivative term on the right hand side of Eq. (13) and find the following evolution equation for the Green's function $G(\phi,\tau)$:

$$\frac{\partial G}{\partial\tau}=2\sigma\frac{\partial^2 G}{\partial\phi^2}-\left[\frac{a}{2}b^2(\phi)+\frac{1}{2}b'(\phi)+\lambda e^{\phi}\right]G, \tag{14}$$

where $a=1/4\sigma$ and $G(\phi,0)=\exp[-1/4\sigma\int_0^{\phi}b(u)du]$ $R(\phi,0)=\sqrt{A}\psi_0(\phi)$, using the Gibbs measure in Eq. (9). To solve Eq. (14) we make the standard eigenvalue decomposition

$$G(\phi,\tau)=\sum_E c_E g_E(\phi)e^{-4\sigma E\tau}, \tag{15}$$

where the eigenfunctions $g_E(\phi)$ satisfy a Schrödinger-like equation,

$$-\frac{1}{2}\frac{d^2 g_E(\phi)}{d\phi^2}+\left[\frac{a^2}{2}b^2(\phi)+\frac{a}{2}b'(\phi)+a\lambda e^{\phi}\right]g_E(\phi)=Eg_E(\phi), \tag{16}$$

with $b(\phi)$ given by Eq. (8) and $a=1/4\sigma$. The coefficients c_E's in Eq. (15) are determined from the initial condition, $G(\phi,0)=\sqrt{A}\psi_0(\phi)$. Using orthogonality of eigenfunctions one finds,

$$c_E=\sqrt{A}\int_{-\infty}^{\infty}d\phi g_E^*(\phi)\psi_0(\phi)=\sqrt{A}\langle g_E|\psi_0\rangle, \tag{17}$$

where we have used the standard bra-ket notation of quantum mechanics. Note also that if we consider the free problem by setting $\lambda=0$, it is easy to verify from Eq. (16) that there is an eigenfunction with energy $E=0$ which corresponds to the stationary state of the free problem. This zero energy eigenfunction is given precisely by $\psi_0(\phi)$ in Eq. (10) and the Gibbs measure is just the square of this eigenfunction, $P_{\rm st}(\phi)=\psi_0^2(\phi)$.

Substituting the c_E's from Eq. (17) in the decomposition equation (15) and setting finally $\tau=x$ we get the following expression of $R(\phi,x)$ in terms of the eigenfunctions:

$$R(\phi,x)=\psi_0^*(\phi)\sum_E\langle g_E|\psi_0\rangle g_E(\phi)e^{-4\sigma Ex}. \tag{18}$$

By integrating $R(\phi,x)$ in Eq. (18) over ϕ, we finally obtain the disorder average $\overline{Q_p(x)}$ in a compact form,

$$\begin{aligned}\overline{Q_p(x)}&=\int_{-\infty}^{\infty}R(\phi,x)d\phi\\&=\int_{-\infty}^{\infty}d\phi\psi_0^*(\phi)g_E(\phi)\sum_E\langle g_E|\psi_0\rangle e^{-4\sigma Ex}\\&=\sum_E\langle g_E|\psi_0\rangle\langle\psi_0|g_E\rangle e^{-4\sigma Ex}\\&=\langle\psi_0|e^{-4\sigma\hat{H}x}|\psi_0\rangle,\end{aligned} \tag{19}$$

where the quantum Hamiltonian $\hat{H}$ in the ϕ basis is given by

$$\hat{H} = -\frac{1}{2}\frac{\partial^2}{\partial \phi^2} + \left[\frac{a^2}{2}b^2(\phi) + \frac{a}{2}b'(\phi) + a\lambda e^{\phi}\right] = \hat{H}_0 + \hat{H}_1. \tag{20}$$

Here $\hat{H}_0 = -\frac{1}{2}\partial^2/\partial\phi^2 + [(a^2/2)b^2(\phi) + (a/2)b'(\phi)]$ is the Hamiltonian of the free problem (corresponding to $\lambda = 0$), $\hat{H}_1 = \lambda a e^{\phi}$ is like a perturbation Hamiltonian and $b(\phi) = -e^{\phi} + 2\mu + 2pe^{-\phi}$.

The exact formula $\overline{Q_p(x)} = \int_0^\infty dt e^{-pt}\overline{F(x,t)} = \langle \psi_0 | e^{-4\sigma \hat{H} x} | \psi_0 \rangle$ in Eq. (19) is, in fact, the central result of this paper. This result tells us that the Laplace transform of the disorder averaged first-passage time can, in principle, be fully computed for an arbitrary starting position x, arbitrary p (and hence for arbitrary t) and also for any value of the drift μ provided one can compute all the eigenvalues and the corresponding eigenfunctions of the Hamiltonian $\hat{H}$ in Eq. (20). In other words, the calculation of the disorder averaged persistence is reduced to finding the spectrum of the quantum Hamiltonian $\hat{H}$. In Secs. III and IV we show how this spectrum can be determined in limiting cases that lead to exact asymptotic results (large t limit) for the persistence $P(x,t)$ for any arbitrary drift μ.

The above derivation based on the formal correspondence between the Fokker-Planck equation and the Schrödinger operator obviously requires some mathematical clarification. The Schrödinger operator in Eq. (20) is a self-adjoint operator with respect to the ordinary L^2 scalar product; whereas the solution of Eq. (13) belongs, in principle, to a different functional space. One can show that the mapping to the Schrödinger problem provides the properly normalized eigenfunctions (at least those belong to the discrete branch of the spectrum); however, the issue of completeness is far from being established [17] (for a related discussion see Ref. [18]). Fortunately in our case there is a strong indication that this mapping is mathematically sound. Several functionals of the Brownian motion have been computed by probabilistic tools and also by an analysis based on the Morse potential [19] and the fact that both methods yield the same answer is a strong indication that this mapping is exact.

III. EXPLICIT RESULTS FOR POSTIVE DRIFT ($\mu > 0$)

In this section we focus on the positive drift ($\mu > 0$) case. We have seen in Sec. II A that for the pure case, the particle eventually escapes to infinity with a nonzero probability when there is a positive drift ($\mu > 0$) away from the origin. This escape probability $P(x)$ is precisely the persistence $P(x,t)$ in the limit $t \to \infty$. Due to a nontrivial dependence of this probability $P(x)$ on the initial position x, we call $P(x)$ the persistence profile. Note that from the relationship $P(x,t) = 1 - \int_0^t F(x,t_*)dt_*$, where $F(x,t)$ is the first-passage time distribution, it follows that the persistence profile is given by $P(x) = 1 - \int_0^\infty F(x,t_*)dt_* = 1 - Q_0(x)$, where we recall that $Q_p(x) = \int_0^\infty e^{-pt}F(x,t)dt$ is just the Laplace transform of the first-passage time distribution.

In the disordered case with positive drift, one would expect a similar behavior; namely for each sample of disorder, the particle will eventually escape to infinity with a nonzero sample dependent probability $P(x)$. The disorder averaged persistence profile is then given by $\overline{P(x)} = 1 - \overline{Q_0(x)}$. Using this relationship one can then compute, in the large t limit, the exact time-independent persistence profile by setting $p = 0$ in the general formula for $\overline{Q_p(x)}$ in Eq. (19) derived in Sec. II B. For $p = 0$, we get from Eq. (8), $b(\phi) = -e^{\phi} + 2\mu$. Substituting this $b(\phi)$ in Eq. (20), setting $\lambda = 1$ and simplifying, we find $\hat{H} = \hat{H}_M + \nu^2/8$ where $\nu = \mu/\sigma$ and $\hat{H}_M$ is a generalized Morse Hamiltonian given by

$$\hat{H}_M = -\frac{1}{2}\frac{\partial^2}{\partial \phi^2} + \frac{a^2}{2}e^{2\phi} - \frac{a(\nu-1)}{2}e^{\phi}. \tag{21}$$

It then follows from Eq. (19) that

$$\overline{Q_0(x)} = e^{-\sigma\nu^2 x/2}\langle \psi_0 | e^{-4\sigma \hat{H}_M x} | \psi_0 \rangle. \tag{22}$$

To evaluate the matrix element in Eq. (22) explicitly we need to know the spectrum of the generalized Morse Hamiltonian $\hat{H}_M$. Fortunately this spectrum can be fully determined. This calculation is done in detail in Appendix A. Here we just summarize this spectrum and use the results to compute $\overline{P(x)} = 1 - \overline{Q_0(x)}$ explicitly.

The spectrum of $\hat{H}_M$ consists of two parts: a discrete part with negative energies corresponding to the bound states and a continuous part with positive energies corresponding to the scattering states (see Appendix A). The nature of the spectrum depends on the parameter $\nu = \mu/\sigma$. It turns out that there is a critical value $\nu_c = 2$ such that for $\nu > \nu_c$ the spectrum has both the bound states and the scattering states. In contrast, for $\nu < \nu_c$, there are no bound states and only scattering states exist. We notice that a similar behavior was obtained in the study of transport properties of the Sinai model [20]. The eigenvalues and the corresponding eigenfunctions are given as follows.

Bound States: The bound states are labeled by an integer n. The eigenvalues are given by

$$E_n = -\tfrac{1}{2}[\nu/2 - 1 - n]^2, \quad n = 0,1,\ldots,[\nu/2 - 1], \tag{23}$$

where $[m]$ indicates the integer part of m. Clearly this discrete spectrum exists provided $\nu > 2$. The corresponding normalized eigenfunctions are given by

$$g_n(\phi) = b_n e^{-\phi/2} W_{(\nu-1)/2,\nu/2-1-n}(2ae^{\phi}), \tag{24}$$

where $W_{\alpha,\beta}(x)$ is the Whittaker function [21]. The normalization constant b_n can also be computed exactly (see Appendix A)

$$b_n^2 = \frac{2\sigma(\nu - 2 - 2n)\Gamma(n+1)}{\Gamma(\nu - 1 - n)}, \tag{25}$$

where $\Gamma(x)$ is the standard Gamma function.

Scattering States: The scattering states have positive energies labeled by the wave vector q, $E_q = q^2/2$ with $0 \leqslant q \leqslant \infty$. The corresponding eigenfunctions are given by

$$g_q(\phi) = b(q) e^{-\phi/2} W_{(\nu-1)/2,iq}(2ae^{\phi}), \tag{26}$$

where the coefficient $b(q)$ is given by

$$b(q)=\frac{1}{\sqrt{2\pi}}(2a)^{-iq-1/2}\frac{\Gamma(1-\nu/2-iq)}{\Gamma(-2iq)}. \qquad (27)$$

This coefficient $b(q)$ is chosen such that in the limit $\phi\to-\infty$ [where the quantum potential in the Hamiltonian $\hat{H}_M$ in Eq. (21) vanishes] the eigenfunction $g_q(\phi)$ approaches a plane wave form, i.e., $g_q(\phi)\to 1/\sqrt{2\pi}[e^{iq\phi}+r(q)e^{-iq\phi}]$ as $\phi\to-\infty$, where $e^{iq\phi}$ represents an incident wave traveling in the direction of the positive ϕ and $e^{-iq\phi}$ represents the reflected wave traveling in the opposite direction with $r(q)$ being the reflection coefficient (for details see Appendix A).

Having obtained the full spectrum of $\hat{H}_M$ we are now ready to compute the persistence profile $\overline{P(x)}=1-\overline{Q_0(x)}$. Expanding the right hand side of Eq. (22) in the energy basis of $\hat{H}_M$ and using the results on the spectrum of $\hat{H}_M$ summarized above, we get

$$\overline{Q_0(x)}=e^{-\sigma\nu^2x/2}\left[\sum_{n=0}^{[\nu/2-1]}|\langle g_n|\psi_0\rangle|^2e^{2\sigma(\nu/2-1-n)^2x}+\int_0^\infty dq|\langle g_q|\psi_0\rangle|^2e^{-2q^2\sigma x}\right], \qquad (28)$$

a result that is valid for all x and for all $\mu>0$. The function $\psi_0(\phi)$ is already known. In fact, for $p=0$, we find from Eqs. (8) and (10) the following normalized expression:

$$\psi_0(\phi)=\frac{1}{\sqrt{\Gamma(\nu)(2\sigma)^\nu}}\exp\left[-\frac{1}{4\sigma}e^\phi+\frac{\nu}{2}\phi\right]. \qquad (29)$$

Using this expression of $\psi_0(\phi)$ and the eigenfunctions in Eqs. (24) and (26) one can easily evaluate the matrix elements $\langle g_n|\psi_0\rangle$ and $\langle g_q|\psi_0\rangle$. For the bound states, we get

$$\langle g_n|\psi_0\rangle=\int_{-\infty}^{\infty}d\phi\psi_0(\phi)g_n(\phi)=\frac{b_n}{\sqrt{2\sigma\Gamma(\nu)}}\Gamma(n+1)\Gamma(\nu-1-n), \qquad (30)$$

where b_n is given by Eq. (25). Similarly for the scattering states, we obtain

$$\langle g_q|\psi_0\rangle=\int_{-\infty}^{\infty}d\phi\psi_0(\phi)g_q(\phi)=\frac{b(q)}{\sqrt{2\sigma\Gamma(\nu)}}\Gamma(\nu/2-iq)\Gamma(\nu/2+iq), \qquad (31)$$

with $b(q)$ given by Eq. (27). Substituting these matrix elements in Eq. (28) we get our final expression,

$$\overline{Q_0(x)}=\frac{e^{-\sigma\nu^2x/2}}{2\sigma\Gamma(\nu)}\left[\sum_{n=0}^{[\nu/2-1]}b_n^2\times\Gamma^2(n+1)\Gamma^2(\nu-1-n)e^{2\sigma(\nu/2-1-n)^2x}+\int_0^\infty dq|b(q)|^2|\Gamma(\nu/2-iq)|^4e^{-2q^2\sigma x}\right], \qquad (32)$$

where b_n and b_q are given, respectively by Eqs. (25) and (27). Substituting Eq. (32) in the relation $\overline{P(x)}=1-\overline{Q_0(x)}$ then gives us the exact persistence profile valid for any $x>0$ and any $\mu>0$.

It is instructive to derive explicitly the tails of this profile $\overline{P(x)}$ for small x and large x. Consider first the limit $x\to 0$. While we can use the general solution in Eq. (32) to derive the small x behavior, it is easier to consider the original equation (19) which for small x gives $\overline{Q_0(x)}\to 1-4\sigma x\langle\psi_0|\hat{H}|\psi_0\rangle$. Since $\hat{H}=\hat{H}_0+\hat{H}_1$ and moreover since $\psi_0(\phi)$ is a zero-energy eigenfunction of $\hat{H}_0$, we get $\overline{Q_0(x)}\to 1-4\sigma x\langle\psi_0|\hat{H}_1|\psi_0\rangle$. Expanding the matrix element in the ϕ basis, using $\hat{H}_1=ae^\phi$ [setting $\lambda=1$ in Eq. (20)] and the expression of $\psi_0(\phi)$ from Eq. (29) and evaluating the resulting integral, we get $\overline{Q_0(x)}\to 1-2\sigma\nu x$. Using $\nu=\mu/\sigma$, we get $\overline{P(x)}=2\mu x$ as $x\to 0$. Thus we obtain an interesting result that the slope 2μ characterizing the linear growth of the profile near $x=0$ is completely independent of the disorder strength σ. In fact, the small x behavior of the disorder averaged persistence profile is identical to that of the pure case.

We now turn to the other limit $x\to\infty$. Here we use Eq. (32). First consider the case when $\nu>2$. Then we know from Eq. (32) that there exist bound states. In that case it is evident that for large x, the term corresponding to the lowest energy bound state ($n=0$) will be the most dominant term on the right hand side of Eq. (32). Retaining only this leading $n=0$ term in Eq. (32) and using $b_0^2=2\sigma/\Gamma(\nu-2)$ we get $\overline{Q_0(x)}=(\nu-2)/(\nu-1)e^{-2\sigma(\nu-1)x}$ as $x\to\infty$ for $\nu>2$. Consider now the opposite case when $\nu<2$. In this case there are no bound states and there is no contribution from the discrete sum on the right hand side of Eq. (32). The only contribution is from the integral representing the scattering states. For large x, the most dominant contribution to the integral will come from the small q regime. Expanding the Γ functions for small q, we find after preliminary algebra, $\overline{Q_0(x)}=A_\nu(2\sigma x)^{-3/2}e^{-\nu^2\sigma x/2}$ as $x\to\infty$ for $\nu<2$, where A_ν is a constant (see below). Exactly at $\nu=2$, we get from Eq. (32), $\overline{Q_0(x)}=e^{-2\sigma x}/\sqrt{2\pi\sigma x}$ for large x. Let us summarize the three different types of large x behaviors of the persistence profile,

$$1-\overline{P(x)}\to\begin{cases}\dfrac{\nu-2}{\nu-1}e^{-2(\nu-1)\sigma x}, & \nu>2\\ \dfrac{1}{\sqrt{2\pi\sigma x}}e^{-2\sigma x}, & \nu=2\\ \dfrac{A_\nu}{(2\sigma x)^{3/2}}e^{-\nu^2\sigma x/2}, & \nu<2,\end{cases} \qquad (33)$$

where $A_\nu = \pi^{3/2}\Gamma^2(\nu/2)/\Gamma(\nu)[1-\cos(\pi\nu)]$. Evidently the shape of the profile in Eq. (33) for large x changes as ν varies through the critical point $\nu_c = 2$. The reason for the existence of this critical point is evident from our analysis. Essentially it happens due to the loss of bound states as ν decreases from $\nu > 2$ to $\nu < 2$. Note that this critical behavior at finite $\nu = \nu_c = 2$ could not be derived by the RSRG method. The RSRG method is valid only in the $\nu \to 0$ limit where the exact result in Eq. (33) coincides with the RSRG results [8].

We conclude this section by pointing out that it is possible to have an alternative derivation of the disorder averaged persistence profile $\overline{P(x)}$ for $\mu > 0$ by a completely different method. This method relies on mapping the calculation of the persistence profile to calculate the disorder average of the ratio of two partition functions in the Sinai model. This mapping makes use of certain mathematical properties of the Brownian motion. The average of this ratio of partition functions was already computed [9,22] in a different context. Using those results one can then recover the results in Eq. (33). The derivation of this mapping is presented in Appendix B.

IV. EXPLICIT RESULTS FOR NEGATIVE DRIFT ($\mu < 0$)

We now turn to the case when $\mu < 0$ which corresponds to a drift towards the origin since the initial position $x > 0$. The situation here is very different from the positive drift $\mu > 0$ case discussed in the preceding section. For $\mu < 0$, we expect from the analogy to the pure case that for each sample, the particle will definitely cross the origin as $t \to \infty$ no matter what the starting position x is. Hence for $\mu < 0$, the persistence $P(x,t) \to 0$ as $t \to \infty$ for all x, unlike the $\mu > 0$ case where the persistence approaches a time independent profile $P(x,t) \to P(x)$ as $t \to \infty$. Therefore what is interesting in the $\mu < 0$ case is to compute the asymptotic behavior of $P(x,t)$ for large t. Recalling the definitions $P(x,t) = 1 - \int_0^t F(x,t_*)dt_*$ and $Q_p(x) = \int_0^\infty e^{-pt}F(x,t)dt$, where $F(x,t)$ is the first-passage time distribution, we see that the analysis of the large but finite t limit of the disorder averaged persistence $\overline{P(x,t)}$ requires an analysis of the $p \to 0$ limit of the Laplace transform $\overline{Q_p(x)}$ [rather than exactly at $p = 0$ where $\overline{Q_0(x)} = 1$ trivially for $\mu < 0$]. In fact we will see later in this section that in this limit $\overline{P(x,t)}$ or equivalently $\overline{F(x,t)}$ display a variety of scaling behaviors as one tunes the relevant parameter $\nu' = -\mu/\sigma$.

As in the case of $\mu > 0$, the starting point of our analysis for $\mu < 0$ is the central result in Eq. (19) which is valid for all μ. Unlike the $\mu > 0$ case, we cannot, however, put $p = 0$ straightway in the Hamiltonian $\hat{H}$ in Eq. (20). To derive the time-dependent asymptotics for large t, we need to analyze the spectrum of $\hat{H}$ keeping p small but *nonzero*. Unfortunately, for nonzero p, it is hard to determine the full spectrum of $\hat{H}$ exactly. Fortunately, however, it is possible to extract the leading asymptotic behavior as $x \to \infty$ and $p \to 0$ without too much trouble. To see this, we consider the energy eigenvalue decomposition in the second line on the right hand side of Eq. (19) and find that as $x \to \infty$ the leading contribution comes from the ground state energy E_0 of $\hat{H}$,

$$\overline{Q_p(x)} \to |\langle \psi_0 | g_0 \rangle|^2 e^{-4\sigma E_0 x}, \tag{34}$$

where $|g_0\rangle$ is the ground state of $\hat{H}$. It is clear from Eq. (34) that to evaluate the large x asymptotics we just need to compute the ground state energy E_0 of the Hamiltonian $\hat{H}$ in Eq. (20). Unfortunately, even the ground state energy E_0 is hard to compute exactly for arbitrary p. One can, however, make progress in the $p \to 0$ limit. To see this we recall that $|\psi_0\rangle$ in Eq. (34) is the exact zero energy eigenstate of the free part of the Hamiltonian $\hat{H}_0$ [see the discussion after Eq. (17)], i.e., $\hat{H}_0|\psi_0\rangle = 0$. Knowing this exact fact, we can then determine the ground state energy E_0 of the full Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_1$ by treating $\hat{H}_1 = \lambda a e^{\phi}$ (where eventually we will set $\lambda = 1$) as a perturbation to the free Hamiltonian $\hat{H}_0$. For example, to first order in the perturbation term, we get $E_0 = 0 + \langle \psi_0 | \hat{H}_1 | \psi_0 \rangle$ where 0 indicates the fact that the ground state energy of the unperturbed Hamiltonian $\hat{H}_0$ is exactly 0. We then decompose this matrix element $\langle \psi_0 | \hat{H}_1 | \psi_0 \rangle$ in the ϕ basis and use $\hat{H}_1 = a e^{\phi}$ (setting $\lambda = 1$) to obtain $E_0 = a \int_{-\infty}^{\infty} \psi_0^2(\phi) e^{\phi} d\phi$. The normalized wave function $\psi_0(\phi)$ is already known from Eq. (10) and has the following explicit expression:

$$\psi_0(\phi) = \sqrt{\frac{(2p)^{\nu'/2}}{2K_{\nu'}(\sqrt{2p}/\sigma)}} \exp\left[-\frac{e^{\phi}}{4\sigma} - \frac{\nu'}{2}\phi - \frac{p}{2\sigma} e^{-\phi} \right], \tag{35}$$

where $K_\alpha(x)$ is the modified Bessel function of index α [21] and we have used the definition $\nu' = -\mu/\sigma$. Using this explicit form of $\psi_0(\phi)$ and carrying out the integral in $E_0 = a \int_{-\infty}^{\infty} \psi_0^2(\phi) e^{\phi} d\phi$, we get, to first order in perturbation theory,

$$E_0 = a\sqrt{2p}\, \frac{K_{1-\nu'}(\sqrt{2p}/\sigma)}{K_{\nu'}(\sqrt{2p}/\sigma)}. \tag{36}$$

The result in Eq. (36) is only up to first order in $\hat{H}_1$. It is not clear, so far, why one should stop at first order only. In other words, we have not yet specified in what sense $\hat{H}_1$ is "small" compared to $\hat{H}_0$. Note that in deriving Eq. (36) we have not yet taken the $p \to 0$ limit. In the limit $p \to 0$, using the asymptotic properties of Bessel functions [21] in Eq. (36), we find

$$E_0 \to \begin{cases} \dfrac{a}{\sigma(\nu'-1)} p, & \nu' > 1 \\ -\dfrac{a}{\sigma} p \ln p & \nu' = 1 \\ a B_{\nu'} p^{\nu'} & 0 < \nu' < 1, \end{cases} \tag{37}$$

where we recall that $a=1/4\sigma$ and the constant $B_{\nu'}$ is given by

$$B_{\nu'}=\frac{2^{1-\nu'}\Gamma(1-\nu')}{\sigma^{2\nu'-1}\Gamma(\nu')}. \tag{38}$$

Thus, in general, $E_0 \sim p^{\alpha}$ as $p\to 0$, where $\alpha=1$ for $\nu' \geqslant 1$ and $\alpha=\nu'$ for $\nu' \leqslant 1$ with additional logarithmic corrections at $\nu'=1$. Hence basically E_0 is small for small p. Naively one would expect that if, indeed, we carry out the perturbation theory in $\hat{H}_1$ to higher orders, the resulting terms will be lower order in p as $p\to 0$. This naive expectation, fortunately, turns out to be true. In fact we show in Appendix C how to estimate the second order term for small p and it turns out to be at least of $O(p^{\alpha+1})$ and hence negligible compared to the first-order term $[\sim O(p^{\alpha})]$ in the $p\to 0$ limit. For small p, this argument therefore justifies in keeping only the first-order term in the perturbation theory in evaluating E_0. Note that the eigenfunction $|g_0\rangle$ also gets modified from the free eigengunction $|\psi_0\rangle$ due to the perturbation term.

Substituting the $p\to 0$ results from Eq. (37) in Eq. (34) for large x and using $a=1/4\sigma$, we then get three different types of scaling behaviors depending on ν',

$$\overline{Q_p(x)} \to \begin{cases} \exp[-px/\sigma(\nu'-1)], & \nu'>1 \\ \exp[p\ln(p)x/\sigma], & \nu'=1 \\ \exp[-B_{\nu'}p^{\nu'}x], & \nu'<1, \end{cases} \tag{39}$$

where $B_{\nu'}$ is given by Eq. (38). Note that to leading order in small p, we need to keep only the zeroth-order term in the amplitude $|\langle\psi_0|g_0\rangle|^2$ of the exponential in Eq. (34). To the zeroth order $|g_0\rangle=|\psi_0\rangle$ and hence to leading order this amplitude is exactly 1 since $|\psi_0\rangle$ is normalized. Thus it is evident from Eq. (39) that for $\nu'>1$, the correct scaling limit is $x\to\infty$, $p\to 0$ but keeping the product px fixed. On the other hand, for $\nu'<1$, the correct scaling combination is $p^{\nu'}x$.

It turns out that the exact asymptotic results in Eq. (39) were also derived recently by mathematicians by using completely different methods that involved rather heavy mathematical machineries. The first derivation is due to Kawazu and Tanaka who used the so called Kotani's formula and Krein's theory of strings [10,11]. However their method did not permit the explicit evaluation of the constant $B_{\nu'}$. More recently, Hu *et al.* [12] presented yet another completely different derivation by mapping the persistence problem with negative drift onto that of a Bessel process and then using some theorems on this Bessel process. Hu *et al.* managed to compute the coefficient $B_{\nu'}$ explicitly. Note, however, that the constant $B_{\nu'}$ in Ref. [12] has an apparently rather different looking form than our expression in Eq. (38) and it requires a bit of work to show that indeed they are exactly identical.

To derive the asymptotic properties of the first-passage time distribution $F(x,t)$ in the time domain, we need to invert the Laplace transform in Eq. (39) with respect to p. For $\nu'>1$, the Laplace inversion is trivial and we get $F(x,t)=\delta(t-x/\sigma(\nu'-1))$ in the scaling limit $x\to\infty$, $t\to\infty$ but keeping the ratio x/t fixed. This result is very similar to the pure case. The δ function indicates that at late times the particle basically moves ballistically with an effective velocity $\sigma(\nu'-1)$. Similarly for $\nu'=1$, by inverting the Laplace tranform we find that in the scaling limit $x\to\infty$, $t\to\infty$ but keeping the ratio $x\ln(x)/t$ fixed, $F(x,t)=\delta(t-x\ln(x)/\sigma)$. The situation becomes somewhat different for $\nu'<1$. In this case the Laplace inversion indicates that in the scaling limit $x\to\infty$, $t\to\infty$; but keeping the ratio $x^{1/\nu'}/t$ fixed, the first-passage time distribution approaches a scaling form $F(x,t)\sim(1/t)f(t/x^{1/\nu'})$. The scaling function $f(y)$ can be formally written in terms of Lévy function $L_{\nu'}(y)$ and we get $f(y)=yL_{\nu'}(y)$. The Lévy function is formally defined by the Bromwich integral (see the Appendix of Ref. [5])

$$L_{\nu'}(y)=\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} ds e^{sy-B_{\nu'}s^{\nu'}}, \tag{40}$$

where d is such that the $s=d$ line in the complex s plane lies to the right of all the singularities (branch cuts) but is otherwise arbitrary.

An explicit expression of the Lévy function can be obtained only for a few special values of ν' [5]. For example, for $\nu'=1/2$, we get $f(y)=yL_{1/2}(y)=e^{-1/2y}/\sqrt{2\pi y}$. This indicates that $F(x,t)=xe^{-x^2/4t}/\sqrt{2\pi t^3}$ and hence the persistence $P(x,t)=1-\int_0^t F(x,t')dt'=\mathrm{erf}(x/\sqrt{2t})$. We thus arrive at an amazing result that for $\nu'=1/2$, i.e., $\mu=-\sigma/2$, the *disorder averaged* persistence has the same asymptotic behavior as the *pure case without drift* ($\mu=0$, $\sigma=0$) (as derived in Sec. II A). A similar coincidence at this special value of $\nu'=1/2$ was also noted in the context of occupation time distribution in the Sinai model [16]. Thus $\mu=-\sigma/2$ seems to be a special line in the $(\mu-\sigma)$ plane where the Sinai model with drift shares the same asymptotic properties as the pure unbiased Brownian motion. Another solvable point is $\nu'=1/3$, where we get $f(y)=yL_{1/3}(y)=(\alpha_0/2\pi\sqrt{y})K_{1/3}(\alpha_0/\sqrt{y})$ with the constant $\alpha_0=4/3^{3/2}[\Gamma(2/3)/\Gamma(1/3)]^{3/2}\sqrt{\sigma}$.

For general $\nu'<1$, while we cannot calculate the scaling function $f(y)$ explicitly, the behaviors at the tails can be easily determined. For example, first consider the limit $y=t/x^{1/\nu'}\to\infty$, i.e., when $t\gg x^{1/\nu'}$. Using the large y behavior of the Lévy function [5] $L_{\nu'}(y)\approx B_{\nu'}\Gamma(1+\nu')\sin(\pi\nu')/\pi y^{\nu'+1}$, we get $F(x,t)\approx\beta_0 x/t^{\nu'+1}$, where $\beta_0=\nu' 2^{1-\nu'}/\Gamma(\nu')\sigma^{2\nu'-1}$. The persistence $P(x,t)=1-\int_0^t F(x,t')dt'$ then behaves in this limit as

$$P(x,t)\approx\frac{2^{1-\nu'}}{\Gamma(\nu')\sigma^{2\nu'-1}}\frac{x}{t^{\nu'}}, \quad t\gg x^{1/\nu'}, \tag{41}$$

indicating a power law decay $P(x,t)\sim t^{-\theta}$ for large t where the persistence exponent $\theta=\nu'$. We now turn to the other tail of the scaling function $f(y)$ when $y=t/x^{1/\nu'}\to 0$, i.e., when $t\ll x^{1/\nu'}$. Using the properties of the Lévy function near $y\to 0$ [5], we get $f(y)=yL_{\nu'}(y)$

$\approx e^{-(1-\nu')/\nu'\zeta}/\sqrt{2\pi(1-\nu')\zeta}$, where $\zeta=(y^{\nu'}/\nu' B_{\nu'})^{1/(1-\nu')}$. This indicates an essential singularity at $y=0$. Using this asymptotic behavior of $F(x,t)$ in the relation $P(x,t)=1-\int_0^t F(x,t')dt'$ we find the following behavior for the persistence:

$$P(x,t)\approx 1-\frac{1}{\sqrt{2\pi\beta_1(1-\nu')}}\left(\frac{x}{t^{\nu'}}\right)^{-1/2(1-\nu')} \times\exp\left[-\frac{1-\nu'}{\nu'}\beta_1\left(\frac{x}{t^{\nu'}}\right)^{1/(1-\nu')}\right], \quad t\ll x^{1/\nu'}, \quad (42)$$

where $\beta_1=(\nu' B_{\nu'})^{1/(1-\nu')}$ and $B_{\nu'}$ is given by Eq. (38).

Let us summarize the main behavior of the disorder averaged persistence for $\nu'<1$. We find that for $\nu'<1$, there exists a single time scale $t_x\sim x^{1/\nu'}$ depending on the initial position x. For $t\gg t_x$, the persistence $P(x,t)$ decays as a power law with an exponent $\theta=\nu'$ and the amplitude of the power law depends on x as in Eq. (41). In the opposite limit when $t\ll t_x$, the persistence drops extremely sharply from its initial value $P(x,0)=1$ as indicated by the essential singularity in the second term on the right hand side of Eq. (42) when $t\ll x^{1/\nu'}$.

Let us conclude this section on negative drift by one final remark. We note that even though we have assumed throughout this section that μ is strictly negative, we can safely take the limit $\mu\to 0^-$ in Eq. (36) which gives $E_0\approx -1/[2\ln p]$ in the limit $p\to 0$. It then follows from Eq. (34) that in the limit of vanishing drift, one gets the asymtotic result $\overline{Q_p(x)}\to 1+(2\sigma/\ln p)x+\ldots$, when $x\ll -1/\ln p$. From this it follows that $\overline{P(x,t)}\approx 2\sigma x/\ln t$ for $\ln t\gg x$, thus recovering the standard Sinai model behavior in the zero drift limit [8,6].

V. SUMMARY AND PERSPECTIVES

In summary, we have obtained exact asymptotic results for the disorder averaged persistence in the Sinai model with an arbitrary drift. Our method maps exactly the problem of computing the persistence to the problem of finding the eigenvalue spectrum of a single-particle quantum Hamiltonian. We have shown that it is possible to find this spectrum in certain cases that allowed us to obtain exact asymptotic results for arbitrary drift. We note that these results could not have been obtained from the existing physical methods (e.g., the RSRG method) that provide exact results only in the limit of zero drift. Our results show that there is a rich variety of asymptotic behaviors in the persistence as one tunes the drift. In particular, the asymptotics undergo interesting ''phase transitions'' at certain critical values of the control parameter ν (the relative strength of the drift over the disorder), e.g., at $\nu=2$, $\nu=0$, and $\nu=-1$. It would be interesting to extend the exact method presented here to calculate other properties in the Sinai model with finite drift, such as the persistence of a thermally averaged trajectory for which the results in the zero drift limit are already known [8,7].

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APPENDIX A: DERIVATION OF THE SPECTRUM OF THE HAMILTONIAN $\hat{H}_M$

In this appendix we derive the spectrum of the generalized Morse Hamiltonian $\hat{H}_M$ given by Eq. (21). We show that the spectrum has a discrete part with negative energies corresponding to the bound states and also a continuous part with positive energies corresponding to scattering states. The eigenfunctions satisfy the Schrödinger-like equation,

$$-\frac{1}{2}\frac{d^2 g_E(\phi)}{d\phi^2}+\left[\frac{a^2}{2}e^{2\phi}-\frac{a(\nu-1)}{2}e^{\phi}\right]g_E(\phi)=E g_E(\phi). \quad (A1)$$

Let us remind the readers that $a=1/4\sigma$. It turns out to be convenient to make a change of variable $y=2ae^{\phi}$. Furthermore, let us substitute $g_E(\phi)=e^{-\phi/2}f(2ae^{\phi})$ in Eq. (A1). Then the function $f(y)$ satisfies the differential equation

$$\frac{d^2 f}{dy^2}+\left[-\frac{1}{4}+\frac{(\nu-1)}{2y}+\frac{1/4+2E}{y^2}\right]f(y)=0, \quad (A2)$$

where we have suppressed the E dependence of the function $f(y)$ for notational convenience. We next consider the negative and the positive part of the spectrum separately.

Bound states: The bound states are in the negative energy part of the spectrum. Let us substitute $E=-\gamma^2/2$ in Eq. (A2) where γ is the eigenvalue to be determined. The second-order differential equation (A2) with $E=-\gamma^2/2$ is known to have two linearly independent solutions $W_{(\nu-1)/2,\gamma}(y)$ and $W_{-(\nu-1)/2,\gamma}(-y)$ where $W_{\alpha,\beta}(x)$ is the Whittaker function [21]. Thus the most general solution of Eq. (A2) can be written as

$$f(y)=D_1 W_{(\nu-1)/2,\gamma}(y)+D_2 W_{-(\nu-1)/2,\gamma}(-y), \quad (A3)$$

where D_1 and D_2 are arbitrary constants. For a large argument, the Whittaker function is known to have the asymptotic behavior [21] $W_{\alpha,\beta}(x)\sim x^{\alpha}e^{-x/2}$. On the other hand the bound states must be normalizable and hence the eigenfunction $g_E(\phi)$ must vanish as $\phi\to\pm\infty$. The vanishing boundary condition at $\phi=\infty$ indicates that the constant $D_2=0$. Note that here we have assumed $\gamma>0$. If $\gamma<0$, then this boundary condition instead sets $D_1=0$. However, the resulting solution is the same. In other words the eigenfunctions corresponding to γ and $-\gamma$ are the same and not linearly independent of each other. Thus without any loss of generality we can assume $\gamma>0$ and set $D_2=0$. Thus the eigenfunction in terms of the original variable is given by

$$g_E(\phi)=D_1 e^{-\phi/2}W_{(\nu-1)/2,\gamma}(2ae^{\phi}). \quad (A4)$$

Note that the eigenvalue γ is yet to be determined. This is done by employing the vanishing boundary condition at the other tail, namely, $g_E(\phi)\to 0$ as $\phi\to-\infty$. Using the small x

behavior of the Whittaker function, $W_{\alpha,\beta}(x) \to \Gamma(2\beta)/\Gamma[1/2+\beta-\alpha]x^{-\beta+1/2}$ as $x\to 0$ in Eq. (A4) as $\phi \to -\infty$, we get

$$g_E(\phi) \approx D_1 (2a)^{-\gamma+1/2} \frac{\Gamma(2\gamma)}{\Gamma(1-\nu/2+\gamma)} e^{-\gamma\phi}. \quad \text{(A5)}$$

Thus the eigenfunction diverges exponentially as $\phi\to-\infty$. The only way the eigenfunction can satisfy the boundary condition $g_E(-\infty)=0$ is if the denominator $\Gamma(1-\nu/2-\gamma)$ in Eq. (A5) is infinite. This happens when the argument of the Gamma function is a negative integer $1-\nu/2-\gamma=-n$ with $n=0,1,\ldots$. This thus fixes the eigenvalue $\gamma=\nu/2-1-n$ with $n=0,1,\ldots$. However, note that the condition $\gamma>0$ indicates that the maximal allowed value for n is $[\nu/2-1]$, where $[x]$ denotes the integer part of x. Thus finally the bound states have discrete eigenvalues $E=-\gamma^2/2=-(\nu/2-1-n)^2/2$ and the corresponding eigenfunctions, labeled by n, are given by

$$g_n(\phi) = b_n e^{-\phi/2} W_{(\nu-1)/2,\nu/2-1-n}(2ae^{\phi}), \quad \text{(A6)}$$

where $b_n=D_1$ is to be fixed from the normalization condition $\int_{-\infty}^{\infty} g_n^2(\phi)d\phi=1$. To perform this integral, we first use the fact that for positive integer n, one can rewrite the Whittaker function in terms of the Laguerre polynomials [21] and then use the following identity (see the Appendix of Ref. [23]):

$$\int_0^{\infty} dx x^{\alpha-1} e^{-x} [L_n^{\alpha}(x)]^2 = \frac{\Gamma(\alpha+n+1)}{\alpha\Gamma(n+1)}. \quad \text{(A7)}$$

One obtains

$$b_n^2 = \frac{2\sigma(\nu-2-2n)\Gamma(n+1)}{\Gamma(\nu-1-n)}. \quad \text{(A8)}$$

Scattering states: We now turn to the positive energy part of the spectrum and set $E=q^2/2$ in Eq. (A2). The resulting differential equation, once again, has two linearly independent solutions $W_{(\nu-1)/2,iq}(y)$ and $W_{-(\nu-1)/2,iq}(-y)$. Note that the function $W_{-(\nu-1)/2,iq}(-y)=W_{-(\nu-1)/2,iq}(-2ae^{\phi}) \sim \exp(e^{\phi})$ as $\phi\to\infty$. Since the eigenfunction $g_E(\phi)$, even though non-normalizable for scattering states, cannot diverge superexponentially as $\phi\to\infty$, this second solution is not allowed. Keeping only the first solution, we get the eigenfuntions, now labeled by q,

$$g_q(\phi) = b(q) e^{-\phi/2} W_{(\nu-1)/2,iq}(2ae^{\phi}). \quad \text{(A9)}$$

The question is how to determine the constant $b(q)$ in Eq. (A9). This is because, unlike the bound states, the eigenfunctions in Eq. (A9) are non-normalizable. To see, this let us examine the behavior of $g_q(\phi)$ near the tail $\phi\to-\infty$, as in the discrete case. Using the asymptotic properties of the Whittaker function, we find that, as $\phi\to-\infty$,

$$g_q(\phi) \to b(q)\left[\frac{\Gamma(-2iq)}{\Gamma(1-\nu/2-iq)}(2a)^{iq+1/2}e^{iq\phi} + \frac{\Gamma(2iq)}{\Gamma(1-\nu/2+iq)}(2a)^{-iq+1/2}e^{-iq\phi}\right]. \quad \text{(A10)}$$

Clearly the $g_q(\phi)$ functions are non-normalizable. Moreover, unlike in the case of bound states where the boundary condition at $\phi=-\infty$ decides the discrete eigenvalues, in this case we have no such condition indicating that all possible values of $q\geqslant 0$ are allowed. Note that, as in the discrete case, $q>0$ and $q<0$ correspond to the same eigenfunction and hence the allowed values of q lie in the range $0\leqslant q\leqslant\infty$. To determine the constant $b(q)$ we note that in the tail $\phi\to-\infty$, the quantum potential in Eq. (A1) vanishes. The resulting differential equation with $E=q^2/2$ allows plane wave solutions of the form

$$g_q(\phi) \approx \frac{1}{\sqrt{2\pi}}[e^{iq\phi}+r(q)e^{-iq\phi}], \quad \text{(A11)}$$

where $e^{iq\phi}$ represents the incoming wave coming from $\phi=-\infty$ and $e^{-iq\phi}$ represents the reflected wave going towards $\phi=-\infty$ with $r(q)$ being the reflection coefficient. The amplitude $1/\sqrt{2\pi}$ ensures that the plane waves $\psi_q(x)=1/\sqrt{2\pi}e^{iqx}$ are properly orthonormalized in the sense that $\langle\psi_q|\psi_q'\rangle=\delta(q-q')$ where $\delta(z)$ is the Dirac delta function. Comparing Eqs. (A10) and (A11) in the regime $\phi\to-\infty$, we determine the constant $b(q)$ up to a phase as

$$b(q) = \frac{1}{\sqrt{2\pi}}(2a)^{-iq-1/2}\frac{\Gamma(1-\nu/2-iq)}{\Gamma(-2iq)}. \quad \text{(A12)}$$

This completes the derivation of the spectrum of the Hamiltonian $\hat{H}_M$.

APPENDIX B: ALTERNATIVE DERIVATION OF THE PERSISTENCE PROFILE FOR $\mu>0$

For each sample of the disorder, the persistence profile $P(x)$ is related to the Laplace transform $Q_p(x)$ of the first-passage time distribution at $p=0$ via the relation $P(x)=1-Q_0(x)$. The quantity $Q_0(x)$ can be obtained exactly by solving the Eq. (2) with $p=0$,

$$Q_0(x) = 1 - \frac{Z_{\mu}(x)}{Z_{\mu}(\infty)}, \quad \text{(B1)}$$

where $Z_{\mu}(x)=\int_0^x e^{2U(x')}dx'$ is the partition function in a finite box of size x with $U(x)=-\int_0^x F(x')dx'=-\mu x+\sqrt{\sigma}B(x)$ being the random potential. It turns out to be useful to rewrite Eq. (B1) in a slightly different form using a well known property of the Brownian motion: If $B(x)$ is a Brownian motion, then $B(x)-B(x')\equiv\tilde{B}(x-x')$ where $\tilde{B}(x)$ is another independent Brownian motion and $\equiv$ indicates that the random variables on both sides have the iden-

tical distribution, though they are not equal. Using this property and after a few steps of elementary algebra, one can rewrite Eq. (B1) as

$$Q_0(x)=1-P(x)=\frac{Z_\mu(\infty)}{Z_\mu(\infty)+\tilde{Z}_{-\mu}(x)}, \tag{B2}$$

where $\tilde{z}_{-\mu}(x)=\int_0^x e^{-2\mu x'+2\tilde{B}(x')}dx'$. Interestingly, exactly the same ratio as in Eq. (B2) has appeared earlier in other contexts and its average (over disorder) is known exactly [9,22]. Using these known results and setting $\nu=\mu/\sigma$, we get exactly the same asymptotic (large x) persistence profile as in Eq. (33), which was obtained in Sec. III using a completely different method.

APPENDIX C: ESTIMATION OF THE ENERGY CHANGE DUE TO THE SECOND-ORDER TERM IN PERTURBATION THEORY FOR $\mu<0$

In this appendix we consider the case $\mu<0$ and provide an estimate of the second-order contribution ΔE_2 to the ground state energy $E=0+\Delta E_1+\Delta E_2$ of the full Hamiltonian $\hat{H}=\hat{H}_0+\hat{H}_1$ given by Eq. (20), treating $\hat{H}_1=ae^{\phi}$ (setting $\lambda=1$) as a perturbation to the unperturbed Hamiltonian $\hat{H}_0=-\frac{1}{2}\partial^2/\partial\phi^2+V_Q(\phi)$. The quantum potential $V_Q(\phi)$ is given explicitly by

$$V_Q(\phi)=\frac{a}{2}\left[ae^{2\phi}+(\nu'-1)e^{\phi}+4a\left(\frac{\nu'^2}{16a^2}-p\right)-2p(\nu'+1)e^{-\phi}+4ap^2e^{-2\phi}\right]. \tag{C1}$$

In writing the explicit form of the potential we have substituted the expression of $b(\phi)$ from Eq. (8) in Eq. (20) and used the definition $a=1/4\sigma$. Let us recall that the ground state energy of the unperturbed Hamiltonian $\hat{H}_0$ is 0 and the ground state wave function is $\psi_0(\phi)$ given explicitly by Eq. (35) is Sec. IV. The first-order contribution $\Delta E_1=\langle\psi_0|\hat{H}_1\psi_0\rangle$ was already evaluated exactly in Eq. (36) and was shown to scale as $\sim p^\alpha$ as $p\to 0$ with $\alpha=1$ for $\nu'\rangle 1$ and $\alpha=\nu'$ for $\nu'<1$. The goal of this appendix is to show that the second-order contribution ΔE_2 is negligible compared to ΔE_1 as $p\to 0$.

Note that for $p>0$, it is clear from Eq. (C1) that $V_Q(\phi)\to\infty$ as $\phi\to\pm\infty$. This indicates that for $p>0$ the spectrum of $\hat{H}_0$ is discrete and consists of bound states only. Let ψ_n's denote the discrete energy eigenfunctions of $\hat{H}_0$ with the corresponding eigenvalues denoted by e_n. The second-order contribution to the ground state ΔE_2 then follows from the standard quantum mechanics,

$$\Delta E_2=\sum_{n>0}\frac{|\langle\psi_0|\hat{H}_1|\psi_n\rangle|^2}{e_0-e_n}=-\sum_{n>0}\frac{\langle\psi_0|\hat{H}_1|\psi_n\rangle|^2}{e_n}, \tag{C2}$$

where we have used the fact that $e_0=0$ as discussed earlier. Using the fact $e_1<e_2<e_3,\ldots,$ in Eq. (C2), one can immediately obtain an upper bound to $-\Delta E_2$,

$$-\Delta E_2\leqslant\frac{1}{e_1}\sum_{n>0}|\langle\psi_0|\hat{H}_1|\psi_n\rangle|^2. \tag{C3}$$

By adding and subtracting the $n=0$ term to the sum on the right hand side of the above inequality and using the completeness of eigenfunctions we get

$$-\Delta E_2\leqslant\frac{1}{e_1}[\langle\psi_0|\hat{H}_1^2|\psi_0\rangle-|\langle\psi_0|\hat{H}_1|\psi_0\rangle|^2]. \tag{C4}$$

The quantity $S=\langle\psi_0|\hat{H}_1^2|\psi_0\rangle-|\langle\psi_0|\hat{H}_1|\psi_0\rangle|^2$ inside the parenthesis on the right hand side of the inequality (C4) can be evaluated exactly. In general, for any m, one can express the matrix element $\langle\psi_0|\hat{H}_1^m|\psi_0\rangle=a^m\int_{-\infty}^{\infty}\psi_0^2(\phi)e^{m\phi}d\phi$ in the ϕ basis by using $\hat{H}_1=ae^{\phi}$. We then substitute the explicit expression of $\psi_0(\phi)$ from Eq. (35). The resulting integral can be performed exactly using the identity [21]

$$\int_0^\infty x^{\nu-1}e^{-\gamma x-\beta/x}dx=2\left(\frac{\beta}{\gamma}\right)^{\nu/2}K_\nu(2\sqrt{(}\beta\gamma). \tag{C5}$$

In fact, throughout this paper, we have heavily used this identity. We then get $\langle\psi_0|\hat{H}_1^m|\psi_0\rangle=a^m(2p)^{m/2}K_{m-\nu'}(\sqrt{2p}/\sigma)/K_{\nu'}(\sqrt{2p}/\sigma)$. The expression of S requires the results for $m=2$ and $m=1$ and we get

$$S(p)=2a^2p\left[\frac{K_{2-\nu'}(\sqrt{2p}/\sigma)}{K_{\nu'}(\sqrt{2p}/\sigma)}-\frac{K_{1-\nu'}^2(\sqrt{2p}/\sigma)}{K_\nu'^2(\sqrt{2p}/\sigma)}\right]. \tag{C6}$$

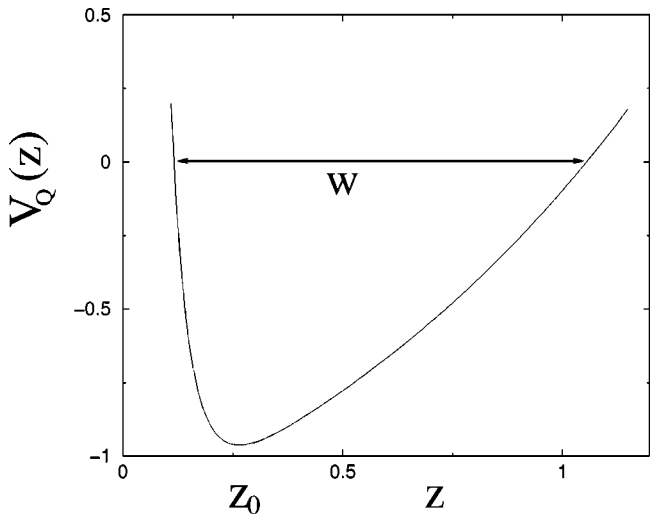

FIG. 2. The shape of the potential $V_Q(z)$ in Eq. (C7) is shown for parameter values: $a=1$, $\nu'=0.5$, and $p=0.1$. The potential has a minimum at $z=z_0\sim p$ and a typical width $W\sim p$ in the limit $p\to 0$.

Using the expansion of Bessel functions for small arguments [21], it is easy to show that in the limit $p \to 0$, $S(p) \sim p^2$ for $\nu' > 2$, $S(p) \to -p^2 \ln p$ for $\nu' = 2$ and $S(p) \sim p^{\nu'}$ for $\nu' < 2$.

Having established the behavior of S for small p, we now need to estimate the gap e_1 (the energy of the first excited state of $\hat{H}_0$) for small p on the right hand side of the inequality $-\Delta E_2 \leqslant S(p)/e_1$ in Eq. (C4). To estimate the gap, we examine the quantum potential in Eq. (C1). It is convenient first to make a change of variable $z = e^{\phi}$ so that $0 \leqslant z \leqslant \infty$. In this new variable the quantum potential in Eq. (C1) reads

$$V_Q(z) = \frac{a}{2}\left[az^2 + (\nu' - 1)z + 4a\left(\frac{\nu'^2}{16a^2} - p \right) - \frac{2p(\nu' + 1)}{z} + \frac{4ap^2}{z^2} \right]. \tag{C7}$$

The shape of this potential is shown in Fig. 2. Note that the potential $V_Q(z)$ has a minimum at $z = z_0$, determined from the equation $dV_Q(z)/dz = 0$, which gives $2az_0^4 + (\nu' - 1)z_0^3 + 2p(\nu' + 1)z_0 - 8ap^2 = 0$. In the limit $p \to 0$, the only real root of this equation is at $z_0 \approx 4ap/(\nu' + 1)$. The value of the potential at this minimum $V_Q(z_0) \to -(2\nu' + 1)/8 + O(p)$ as $p \to 0$. We also need to estimate the typical width of the potential $W(e)$ at an energy e for small p (see Fig. 2). The points $z_{\pm}(e)$, where $V_Q(z) = e$, can be easily estimated for small p since in this limit one just needs to solve a quadratic equation and we get

$$z_{\pm}(e) \approx \frac{4a}{[\nu' + 1 \mp \sqrt{2\nu' + 1 + 2e}]} p + O(p^2). \tag{C8}$$

Hence the typical width of the potential at energy e scales as $W(e) \approx z_+(e) - z_-(e) \sim p$ in the $p \to 0$ limit at any finite level e. For small p, one can approximate the potential $V_Q(z)$ around its minimum $z = z_0$ by a harmonic oscillator potential, $V_q(z) \approx -(2\nu' + 1)/8 + \omega^2 z^2/2$ where the frequency ω is estimated from the typical width, i.e., $\omega^2 W^2/2 \sim O(1)$. Since $W(e) \sim p$, we find that the frequency scales as $\omega \sim 1/p$ as $p \to 0$. One knows that the gap between the first excited state and the ground state in a harmonic potential scales as $e_1 \sim \omega$. Thus we estimate that the energy of the first excited state scales as $e_1 \sim \omega \sim 1/p$ in the limit $p \to 0$.

Substituting this estimate of the gap in the inequality $-\Delta E_2 \leqslant S(p)/e_1$ and using the small p estimates of $S(p)$ derived earlier, we find that as $p \to 0$, $-\Delta E_2 \leqslant p^3$ for $\nu' > 2$, $-\Delta E_2 \leqslant -p^3 \ln p$ for $\nu' = 2$ and $-\Delta E_2 \leqslant p^{\nu'+1}$ for $\nu' < 1$. Comparing these results with the first-order contribution, where $\Delta E_1 \sim p$ for $\nu' > 1$ and $\Delta E_1 \sim p^{\nu'}$ for $\nu' < 1$, we conclude that the second-order contribution is negligible compared to the first-order term in the limit $p \to 0$.

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